

DUAL PERFECT BASES AND DUAL PERFECT GRAPHS

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Dedicated to Professor Boris Feigin on the occasion of his sixtieth birthday

ABSTRACT. We introduce the notion of dual perfect bases and dual perfect graphs. We show that every integrable highest weight module $V_q(\lambda)$ over a quantum generalized Kac-Moody algebra $U_q(\mathfrak{g})$ has a dual perfect basis and its dual perfect graph is isomorphic to the crystal $B(\lambda)$. We also show that the negative half $U_q^-(\mathfrak{g})$ has a dual perfect basis whose dual perfect graph is isomorphic to the crystal $B(\infty)$. More generally, we prove that all the dual perfect graphs of a given dual perfect space are isomorphic as abstract crystals. Finally, we show that the isomorphism classes of finitely generated graded projective indecomposable modules over a Khovanov-Lauda-Rouquier algebra and its cyclotomic quotients form dual perfect bases for their Grothendieck groups.

INTRODUCTION

In [1], Berenstein and Kazhdan introduced the notion of *perfect bases* for integrable highest weight modules over Kac-Moody algebras. Using the properties of perfect bases, they obtained Kashiwara's crystal structure without taking quantum deformation and crystal limits.

Their work was extended by Kang, Oh and Park to the integrable highest weight modules $V_q(\lambda)$ ($\lambda \in P^+$) over a quantum generalized Kac-Moody algebra $U_q(\mathfrak{g})$ and to the negative half $U_q^-(\mathfrak{g})$ [7, 8]. It was shown that the upper global bases (or dual canonical bases) $\mathbb{B}(\lambda)$ and $\mathbb{B}(\infty)$ are perfect bases of $V_q(\lambda)$ and $U_q^-(\mathfrak{g})$, respectively. They also showed that all the crystals arising from perfect bases of $V_q(\lambda)$ and $U_q^-(\mathfrak{g})$ are isomorphic to the crystals $B(\lambda)$ and $B(\infty)$, respectively.

The perfect basis theory plays an important role in the categorification of quantum generalized Kac-Moody algebras. To be more precise, let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ and let $U_{\mathbb{A}}^-(\mathfrak{g})$ be the integral form of $U_q^-(\mathfrak{g})$. Let R be the Khovanov-Lauda-Rouquier algebra associated with the Borchers-Cartan datum for $U_q(\mathfrak{g})$. We denote by $\text{Rep}(R)$ the category of finite-dimensional graded R -modules and let $[\text{Rep}(R)]$ denote its

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Grothendieck group. Then it was proved that $[\text{Rep}(R)]$ is isomorphic to $U_{\mathbb{A}}^-(\mathfrak{g})^\vee$, the dual of $U_{\mathbb{A}}^-(\mathfrak{g})$ with respect to a non-degenerate symmetric bilinear form on $U_q^-(\mathfrak{g})$ [8, 11, 12, 14]. Moreover, in [8] (see also [13]), it was shown that the isomorphism classes of finite-dimensional graded irreducible R -modules form a perfect basis \mathbb{B} of $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R)]$. Thus \mathbb{B} has a crystal structure which is isomorphic to $B(\infty)$.

Similarly, the cyclotomic Khovanov-Lauda-Rouquier algebra R^λ ($\lambda \in P^+$) provides a categorification of $V_q(\lambda)$ in the sense that $[\text{Rep}(R^\lambda)]$ is isomorphic to $V_{\mathbb{A}}(\lambda)^\vee$, the dual of the integral form $V_{\mathbb{A}}(\lambda)$ of $V_q(\lambda)$ ([4, 5]). As in the case with $U_q^-(\mathfrak{g})$, the isomorphism classes of finite-dimensional graded irreducible R^λ -modules form a perfect basis \mathbb{B}^λ of $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R^\lambda)]$ and \mathbb{B}^λ has a crystal structure which is isomorphic to $B(\lambda)$.

On the other hand, let $\text{Proj}(R)$ (respectively, $\text{Proj}(R^\lambda)$) be the category of finitely generated graded projective R -modules (respectively, R^λ -modules). Then they also provide a categorification of $U_q^-(\mathfrak{g})$ and $V_q(\lambda)$. That is, in [4, 5, 11, 12, 14], it was shown that $[\text{Proj}(R)]$ (respectively, $[\text{Proj}(R^\lambda)]$) is isomorphic to $U_{\mathbb{A}}^-(\mathfrak{g})$ (respectively, $V_{\mathbb{A}}(\lambda)$). Note that the isomorphism classes of finitely generated graded projective indecomposable modules form a basis of $[\text{Proj}(R)]$ and $[\text{Proj}(R^\lambda)]$, respectively. To describe their properties, we need the dual notion of perfect bases.

When the Cartan datum is symmetric, the isomorphism classes of finite-dimensional graded irreducible modules correspond to Kashiwara's upper global basis and the isomorphism classes of finitely generated graded projective indecomposable modules correspond to Kashiwara's lower global basis (or Lusztig's canonical basis) under the categorification ([6, 15, 16]). However, when the Cartan datum is not symmetric, the above statement is not true in general.

Thus it is an interesting problem to characterize the bases of $U_q^-(\mathfrak{g})$ and $V_q(\lambda)$ that correspond to the isomorphism classes of finite-dimensional graded irreducible modules and finitely generated graded projective indecomposable modules over R and R^λ .

In this paper, as the first step toward this direction, we introduce the notion of *dual perfect bases* and *dual perfect graphs*. The typical examples of dual perfect bases are the lower global bases $\mathbf{B}(\infty)$ and $\mathbf{B}(\lambda)$ of $U_q^-(\mathfrak{g})$ and $V_q(\lambda)$, respectively. It is straightforward to verify that their dual perfect graphs are isomorphic to the crystals $B(\infty)$ and $B(\lambda)$.

More generally, we show that all the dual perfect graphs of a given dual perfect space are isomorphic as abstract crystals. Thus every dual perfect graph of $U_q^-(\mathfrak{g})$ is isomorphic to $B(\infty)$ and the same statement holds for $V_q(\lambda)$.

Finally, we show that the dual basis of a perfect basis is a dual perfect basis. Therefore, the isomorphism classes of finitely generated graded projective indecomposable modules form a dual perfect basis of $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R)]$ and $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R^\lambda)]$, respectively.

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1. QUANTUM GENERALIZED KAC-MOODY ALGEBRAS

We first recall the basic theory of quantum generalized Kac-Moody algebras. Let I be an index set. An integral square matrix $A = (a_{ij})_{i,j \in I}$ is called a *symmetrizable Borcherds-Cartan matrix* if (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ for $i \neq j$, (iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$, (iv) there is a diagonal matrix $D = \text{diag}(s_i \in \mathbb{Z}_{>0})_{i \in I}$ such that DA is symmetric.

An index i is *real* if $a_{ii} = 2$ and is *imaginary* if $a_{ii} \leq 0$. We write $I^{\text{re}} := \{i \in I \mid a_{ii} = 2\}$ and $I^{\text{im}} := \{i \in I \mid a_{ii} \leq 0\}$. In this paper, we assume that

$$a_{ii} \in 2\mathbb{Z} \quad \text{for all } i \in I.$$

A quadruple (A, P, Π, Π^\vee) consisting of

- (a) a symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
- (b) a free abelian group P , which is called the *weight lattice*,
- (c) the set of simple roots $\Pi = \{\alpha_i \in P \mid i \in I\}$,
- (d) the set of simple coroots $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$

is called a *Borcherds-Cartan datum* if it satisfies the following properties:

- (1) $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$,
- (2) for any $i, j \in I$, there is $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$,
- (3) Π is a linearly independent set.

The subset $P^+ := \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, i \in I\} \subset P$ is called the set of dominant integral weights. We denote by $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ the root lattice and denote by $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ the positive root lattice. We also call $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$ the Cartan subalgebra.

We use the notation

$$(1.1) \quad [n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [m]_i! := [m]_i[m-1]_i \cdots [1]_i, \quad \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}_i := \frac{[m_1]_i!}{[m_2]_i![m_1 - m_2]_i!},$$

where $q_i = q^{s_i}$ for $i \in I$ and $[0]_q! := 1$.

Definition 1.1. The *quantum generalized Kac-Moody algebra* $U_q(\mathfrak{g})$ associated with a Borcherds-Cartan datum (A, P, Π, Π^\vee) is the associative algebra over $\mathbb{Q}(q)$ with unity generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) subject to the following defining relations:

- (1) $q^0 = 1, q^h q^{h'} = q^{h+h'}$,
- (2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,
- (3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q^{s_i h_i}$,

$$(4) \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0 \text{ if } a_{ii} = 2 \text{ and } i \neq j,$$

$$(5) \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \text{ if } a_{ii} = 2 \text{ and } i \neq j,$$

$$(6) e_i e_j - e_j e_i = 0, f_i f_j - f_j f_i = 0 \text{ if } a_{ij} = 0.$$

For $k \in \mathbb{Z}_{>0}$, let

$$e_i^{(k)} = \begin{cases} \frac{e_i^k}{[k]_i!} & \text{if } i \in I^{\text{re}}, \\ e_i^k & \text{if } i \in I^{\text{im}}, \end{cases} \quad f_i^{(k)} = \begin{cases} \frac{f_i^k}{[k]_i!} & \text{if } i \in I^{\text{re}}, \\ f_i^k & \text{if } i \in I^{\text{im}}. \end{cases}$$

Let $U_q^-(\mathfrak{g})$ and $U_q^+(\mathfrak{g})$ be subalgebras of $U_q(\mathfrak{g})$ generated by f_i 's and e_i 's ($i \in I$), respectively, and let U_q^0 be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h 's ($h \in P^\vee$). Then the algebra $U_q(\mathfrak{g})$ has the triangular decomposition:

$$U_q(\mathfrak{g}) = U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$

Fix $i \in I$. For any $P \in U_q^-(\mathfrak{g})$, there exist unique elements $Q, R \in U_q^-(\mathfrak{g})$ such that

$$e_i P - P e_i = \frac{K_i Q - K_i^{-1} R}{q_i - q_i^{-1}}.$$

We define the endomorphisms $e'_i, e''_i: U_q^-(\mathfrak{g}) \rightarrow U_q^-(\mathfrak{g})$ by

$$e'_i(P) = R, \quad e''_i(P) = Q.$$

Then we have the following commutation relations:

$$e'_i f_j = \delta_{ij} + q_i^{-a_{ij}} f_j e'_i \text{ for } i, j \in I.$$

As we can see in [3, 9], there exists a unique non-degenerate symmetric bilinear form (\cdot, \cdot) on $U_q^-(\mathfrak{g})$ satisfying

$$(1.2) \quad (1, 1) = 1, \quad (f_i P, Q) = (P, e'_i Q) \text{ for all } P, Q \in U_q^-(\mathfrak{g}).$$

We now turn to the representation theory of $U_q(\mathfrak{g})$. A $U_q(\mathfrak{g})$ -module V is called a *highest weight module* with *highest weight* λ if there exists a nonzero vector $v_\lambda \in V$ such that

$$(1) e_i v_\lambda = 0 \text{ for } i \in I, \quad (2) q^h v_\lambda = q^{\lambda(h)} v_\lambda \text{ for } h \in P^\vee, \quad (3) U_q(\mathfrak{g}) v_\lambda = V.$$

The vector v_λ is called a highest weight vector. For each $\lambda \in P$, there exists a unique irreducible highest weight module $V_q(\lambda)$ up to an isomorphism.

Consider the anti-involution ϕ on $U_q(\mathfrak{g})$ given by

$$q^h \mapsto q^h, \quad e_i \mapsto f_i, \quad f_i \mapsto e_i \text{ for } i \in I, h \in P^\vee.$$

Then there exists a unique non-degenerate symmetric bilinear form $(\ , \)$ on $V_q(\lambda)$ given by

$$(1.3) \quad (v_\lambda, v_\lambda) = 1, \quad (Pu, v) = (u, \phi(P)v) \quad \text{for all } P \in U_q(\mathfrak{g}) \text{ and } u, v \in V_q(\lambda).$$

Definition 1.2. The category \mathcal{O}_{int} consists of $U_q(\mathfrak{g})$ -modules V satisfying the following properties:

(a) V has a weight space decomposition, i.e., $V = \bigoplus_{\mu \in P} V_\mu$, where

$$V_\mu = \{v \in V \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\},$$

(b) there exist a finite number of weights $\lambda_1, \dots, \lambda_t \in P$ such that

$$\text{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\} \subset \bigcup_{j=1}^t (\lambda_j - Q^+),$$

(c) if $a_{ii} = 2$, then the action of f_i on V is locally nilpotent,

(d) if $a_{ii} \leq 0$, then $\mu(h_i) \in \mathbb{Z}_{\geq 0}$ for every $\mu \in \text{wt}(V)$,

(e) if $a_{ii} \leq 0$ and $\mu(h_i) = 0$, then $f_i(V_\mu) = 0$,

(f) if $a_{ii} \leq 0$ and $\mu(h_i) = -a_{ii}$, then $e_i(V_\mu) = 0$.

The following proposition was proved in [3].

Proposition 1.3 ([3]).

(i) *The category \mathcal{O}_{int} is semisimple.*

(ii) *If $\lambda \in P^+$, then $V_q(\lambda)$ is a simple object of \mathcal{O}_{int} .*

(iii) *Every simple object of the category \mathcal{O}_{int} has the form $V_q(\lambda)$ for some $\lambda \in P^+$.*

2. LOWER CRYSTAL BASES

Let V be a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} . It is straightforward to verify that every vector $v \in V$ can be uniquely written as

$$v = \sum_{k \geq 0} f_i^{(k)} v_k,$$

where $v_k \in \ker e_i$ with $v_k = 0$ for $k \gg 0$. We define the *Kashiwara operators* \tilde{e}_i and \tilde{f}_i ($i \in I$) by

$$\tilde{e}_i v = \sum_{k=1}^N f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k=0}^N f_i^{(k+1)} v_k.$$

Let

$$\mathbb{A}_0 := \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}.$$

Definition 2.1. A free \mathbb{A}_0 -submodule L of V is called a *lower crystal lattice* of V if it satisfies

- (1) $\mathbb{Q}(q) \otimes_{\mathbb{A}_0} L = V$,
- (2) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap V_\lambda$,
- (3) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$.

Definition 2.2. Let L be a lower crystal lattice of V and let B be a \mathbb{Q} -basis of L/qL . A pair (L, B) is called a *lower crystal basis* of V if it satisfies

- (1) $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
- (2) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$,
- (3) for any $b, b' \in B$ and $i \in I$, we have $\tilde{f}_i(b) = b'$ if and only if $\tilde{e}_i b' = b$.

We define the I -colored arrows on B by setting $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i(b) = b'$. The I -colored oriented graph (B, \rightarrow) thus defined is called the *crystal graph* or simply the *crystal* of V .

It is known that every $U_q(\mathfrak{g})$ -module in \mathcal{O}_{int} has a lower crystal basis.

Proposition 2.3 ([3, 9]). *Let $V_q(\lambda)$ ($\lambda \in P^+$) be the irreducible highest weight module in the category \mathcal{O}_{int} with highest vector v_λ . Then $V(\lambda)$ has a unique lower crystal basis $(L(\lambda), B(\lambda))$, where*

$$(2.1) \quad \begin{aligned} L(\lambda) &= \mathbb{A}_0\text{-submodule generated by } \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} v_\lambda \mid i_s \in I, k \geq 0\}, \\ B(\lambda) &= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} v_\lambda \bmod qL(\lambda) \mid i_s \in I, k \geq 0\} \setminus \{0\} \subset L(\lambda)/qL(\lambda). \end{aligned}$$

Similarly, using the operator e'_i in place of e_i , we can develop the lower crystal basis theory for $U_q^-(\mathfrak{g})$. In particular, we have the following proposition.

Proposition 2.4 ([3, 9]). *Let*

$$\begin{aligned} L(\infty) &= \mathbb{A}_0\text{-submodule generated by } \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} 1 \mid i_s \in I, k \geq 0\}, \\ B(\infty) &= \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} 1 \bmod qL(\infty) \mid i_s \in I, k \geq 0\} \subset L(\infty)/qL(\infty). \end{aligned}$$

Then the pair $(L(\infty), B(\infty))$ is a unique lower crystal basis of $U_q^-(\mathfrak{g})$.

Motivated by the properties of lower crystal bases, we define the notion of *abstract crystals* and *crystal morphisms* as follows.

Definition 2.5. An *abstract crystal* is a set B together with the maps $\text{wt}: B \rightarrow P$, $\varphi_i, \varepsilon_i: B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i: B \rightarrow B \cup \{0\}$ with the following properties:

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$.
- (2) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$.
- (3) $b = \tilde{e}_i b'$ if and only if $\tilde{f}_i b = b'$, where $b, b' \in B$, $i \in I$.
- (4) If $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

(5) If $b \in B$ and $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \begin{cases} \varepsilon_i(b) - 1 & \text{if } i \in I^{\text{re}}, \\ \varepsilon_i(b) & \text{if } i \in I^{\text{im}}, \end{cases} \quad \varphi_i(\tilde{e}_i b) = \begin{cases} \varphi_i(b) + 1 & \text{if } i \in I^{\text{re}}, \\ \varphi_i(b) + a_{ii} & \text{if } i \in I^{\text{im}}. \end{cases}$$

(6) If $b \in B$ and $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \begin{cases} \varepsilon_i(b) + 1 & \text{if } i \in I^{\text{re}}, \\ \varepsilon_i(b) & \text{if } i \in I^{\text{im}}, \end{cases} \quad \varphi_i(\tilde{f}_i b) = \begin{cases} \varphi_i(b) - 1 & \text{if } i \in I^{\text{re}}, \\ \varphi_i(b) - a_{ii} & \text{if } i \in I^{\text{im}}. \end{cases}$$

Definition 2.6. Let B_1 and B_2 be abstract crystals. A *crystal morphism* between B_1 and B_2 is a map $\tilde{\psi} : B_1 \rightarrow B_2 \sqcup \{0\}$ with the following properties:

- (1) For $b \in B_1$, $\tilde{\psi}(b) \in B_2$ and $i \in I$, we have $\text{wt}(\tilde{\psi}(b)) = \text{wt}(b)$, $\varepsilon_i(\tilde{\psi}(b)) = \varepsilon_i(b)$ and $\varphi_i(\tilde{\psi}(b)) = \varphi_i(b)$.
- (2) Suppose $b, b' \in B_1$ and $\tilde{\psi}(b), \tilde{\psi}(b') \in B_2$. If $\tilde{f}_i b = b'$, then $\tilde{f}_i(\tilde{\psi}(b)) = \tilde{\psi}(b')$ and $\tilde{e}_i(\tilde{\psi}(b')) = \tilde{\psi}(b)$ for $i \in I$.

3. LOWER GLOBAL BASES

Let V be a vector space over $\mathbb{Q}(q)$ and let

$$\mathbb{A} := \mathbb{Q}[q, q^{-1}], \quad \mathbb{A}_\infty := \{f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = \infty\}.$$

Let $L_{\mathbb{A}}$ (respectively, L_0 and L_∞) be a free \mathbb{A} -submodule (respectively, free \mathbb{A}_0 -submodule and free \mathbb{A}_∞ -submodule) of V such that

$$\mathbb{Q}(q) \otimes_{\mathbb{A}} L_{\mathbb{A}} \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L_0 \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_\infty} L_\infty \cong V.$$

Definition 3.1.

- (1) The triple $(L_{\mathbb{A}}, L_0, L_\infty)$ is called a *balanced triple* of V if the canonical projection $\pi : L_{\mathbb{A}} \cap L_0 \cap L_\infty \rightarrow L_0/qL_0$ is an isomorphism.
- (2) Assume that $(L_{\mathbb{A}}, L_0, L_\infty)$ is a balanced triple. Let B be a \mathbb{Q} -basis of L_0/qL_0 and let $G : L_0/qL_0 \rightarrow L_{\mathbb{A}} \cap L_0 \cap L_\infty$ be the inverse of π . Then the set $\mathbf{B} := \{G(b) \mid b \in B\}$ is called the *lower global basis* of V corresponding to B .

Consider the \mathbb{Q} -algebra automorphism $- : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by

$$\overline{q} = q^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h}.$$

Define an involution $-$ of $V_q(\lambda)$ by

$$\overline{Pv_\lambda} = \overline{P}v_\lambda,$$

where $P \in U_q(\mathfrak{g})$ and $v_\lambda \in V_q(\lambda)$ is the highest weight vector.

Let $(L(\lambda), B(\lambda))$ be the lower crystal basis of $V_q(\lambda)$ given in Proposition 2.3 and let $U_{\mathbb{A}}^-(\mathfrak{g})$ be the \mathbb{A} -subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_i^{(n)}$ ($i \in I, n \geq 0$). Set

$$V_{\mathbb{A}}(\lambda) := U_{\mathbb{A}}^-(\mathfrak{g})v_{\lambda}.$$

Then $(V_{\mathbb{A}}(\lambda), L(\lambda), \overline{L(\lambda)})$ is a balanced triple of $V_q(\lambda)$. Since $B(\lambda)$ is a \mathbb{Q} -basis of $L(\lambda)/qL(\lambda)$, we obtain the lower global basis $\mathbf{B}(\lambda) = \{G(b) \mid b \in B(\lambda)\}$ of $V_q(\lambda)$.

Similarly, $(U_{\mathbb{A}}^-(\mathfrak{g}), L(\infty), \overline{L(\infty)})$ is a balanced triple of $U_q^-(\mathfrak{g})$ and we get the lower global basis $\mathbf{B}(\infty) = \{G(b) \mid b \in B(\infty)\}$ of $U_q^-(\mathfrak{g})$.

The lower global bases satisfy the following properties.

Proposition 3.2 ([2, 3, 9]).

- (i) For any $b \in B(\lambda)$ with $\lambda \in P^+$, $G(b)$ is a unique element in $V_{\mathbb{A}}(\lambda) \cap L(\lambda)$ such that

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \pmod{qL(\lambda)}.$$

- (ii) For any $b \in B(\infty)$, $G(b)$ is a unique element in $U_{\mathbb{A}}^-(\mathfrak{g}) \cap L(\infty)$ such that

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \pmod{qL(\infty)}.$$

Proposition 3.3 ([10]). The lower global bases $\mathbf{B}(\lambda)$ and $\mathbf{B}(\infty)$ satisfy the following properties.

- (i) For any $n \in \mathbb{Z}_{\geq 0}$ and $b \in B(\lambda)$ (respectively, $b \in B(\infty)$), the subset $\{G(b) \mid \varepsilon_i(b) \geq n\}$ of $\mathbf{B}(\lambda)$ (respectively, of $\mathbf{B}(\infty)$) is an \mathbb{A} -basis of $\sum_{k \geq n} f_i^{(k)} V_{\mathbb{A}}(\lambda)$ (respectively, of $\sum_{k \geq n} f_i^{(k)} U_{\mathbb{A}}^-(\mathfrak{g})$).
- (ii) For any $i \in I$ and $b \in B(\lambda)$ (respectively, $b \in B(\infty)$), we have

$$f_i G(b) = \begin{cases} [1 + \varepsilon_i(b)]_i G(\tilde{f}_i b) + \sum_{b'} F_{b, b'}^i G(b') & \text{if } i \in I^{\text{re}}, \\ G(\tilde{f}_i b) & \text{if } i \in I^{\text{im}}, \end{cases}$$

where the sum ranges over b' such that $\varepsilon_i(b') > 1 + \varepsilon_i(b)$ and $F_{b, b'}^i \in \mathbb{A}$.

4. DUAL PERFECT BASES

In this section, we introduce the notion of *dual perfect bases*. Fix a Borcherds-Cartan datum (A, P, Π, Π^\vee) and let \mathbf{k} be a field.

Definition 4.1. Let $V = \bigoplus_{\mu \in P} V_{\mu}$ be a P -graded \mathbf{k} -vector space and let $\{f_i\}_{i \in I}$ be a family of endomorphisms of V . We say that $(V, \{f_i\}_{i \in I})$ is a *pre-dual perfect space* if it satisfies the following conditions.

- (1) There exist finitely many elements $\lambda_1, \dots, \lambda_k \in P$ such that $\text{wt}(V) \subset \bigcup_{j=1}^k (\lambda_j - Q^+)$.
- (2) $f_i(V_{\mu}) \subset V_{\mu - \alpha_i}$ for any $i \in I$ and $\mu \in P$.

For each $i \in I$ and $v \in V \setminus \{0\}$, define $\ell_i(v)$ to be the non-negative integer n such that $v \in f_i^n V \setminus f_i^{n+1} V$.

Definition 4.2. Let $(V, \{f_i\}_{i \in I})$ be a pre-dual perfect space.

- (i) A basis \mathbf{B} of V is called a *dual perfect basis* if
 - (a) $\mathbf{B} = \bigsqcup_{\mu \in P} \mathbf{B}_\mu$, where $\mathbf{B}_\mu = \mathbf{B} \cap V_\mu$,
 - (b) For any $i \in I$, there exists a map $\mathbf{f}_i: \mathbf{B} \rightarrow \mathbf{B} \cup \{0\}$ such that for any $b \in \mathbf{B}$, there exists $c \in \mathbf{k}^\times$ satisfying

$$f_i(b) - c \mathbf{f}_i(b) \in f_i^{\ell_i(b)+2} V.$$

- (c) If $\mathbf{f}_i(b) = \mathbf{f}_i(b') \neq 0$, then $b = b'$.
- (ii) V is called a *dual perfect space* if it has a dual perfect basis.

Proposition 4.3. Every $U_q(\mathfrak{g})$ -module in \mathcal{O}_{int} has a dual perfect basis.

Proof. It suffices to show that every irreducible highest weight module $V_q(\lambda)$ ($\lambda \in P^+$) has a dual perfect basis. Let $\mathbf{B}(\lambda) = \{G(b) \mid b \in B(\lambda)\}$ be the lower global basis of $V_q(\lambda)$. Define

$$\mathbf{f}_i G(b) := G(\tilde{f}_i b) \text{ for } b \in B(\lambda).$$

Then by Proposition 3.3, $\mathbf{B}(\lambda)$ is a dual perfect basis. □

Let \mathbf{B} be a dual perfect basis of a P -graded \mathbf{k} -vector space V .

Lemma 4.4. Let $i \in I$.

- (i) For any $b \in \mathbf{B}$ and $n \in \mathbb{Z}_{\geq 0}$, there exists $c \in \mathbf{k}^\times$ such that
- $$(4.1) \quad f_i^n b - c \mathbf{f}_i^n(b) \in f_i^{n+\ell_i(b)+1} V.$$
- (ii) For any $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$f_i^n V = \bigoplus_{b \in \mathbf{f}_i^n \mathbf{B}} \mathbf{k} b.$$

- (iii) For any $b \in \mathbf{B}$, we have

$$\ell_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid b \in \mathbf{f}_i^n \mathbf{B}\}.$$

- (iv) For any $b \in \mathbf{B}$ such that $\mathbf{f}_i b \neq 0$, we have

$$\ell_i(\mathbf{f}_i b) = \ell_i(b) + 1.$$

- (v) For any $n \in \mathbb{Z}_{\geq 0}$, the image of $\{b \in \mathbf{B} \mid \ell_i(b) = n\}$ is a basis of $f_i^n V / f_i^{n+1} V$.

Proof. (i) By the definition, we have $\ell_i(\mathbf{f}_i b) \geq \ell_i(b) + 1$ for any $b \in \mathbf{B}$ such that $\mathbf{f}_i b \in \mathbf{B}$. Hence we have

$$\ell_i(\mathbf{f}_i^n b) \geq \ell_i(b) + n \text{ for any } b \in \mathbf{B} \text{ and } n \in \mathbb{Z}_{\geq 0} \text{ such that } \mathbf{f}_i^n b \in \mathbf{B}.$$

We shall show (i) by induction on n . If $n = 0, 1$, (i) is obvious. Assume $n > 1$. Then the induction hypothesis implies that $f_i^{n-1}b - cf_i^{n-1}b \in f_i^{n+\ell_i(b)}V$ for some $c \in \mathbf{k}^\times$. Therefore, we have $f_i^n b - cf_i \mathbf{f}_i^{n-1}b \in f_i^{n+\ell_i(b)+1}V$. Hence, if $\mathbf{f}_i^{n-1}b = 0$, then we obtain (4.1). If $\mathbf{f}_i^{n-1}b \in \mathbf{B}$, then we have $f_i \mathbf{f}_i^{n-1}b - c' \mathbf{f}_i^n b \in f_i^{\ell_i(\mathbf{f}_i^{n-1}b)+2}V \subset f_i^{n-1+\ell_i(b)+2}V$ for some $c' \in \mathbf{k}^\times$. Hence we obtain (i).

(ii) By (i), we have $\mathbf{f}_i^n b \in f_i^n V$. Hence it is enough to show that $f_i^n V \subset \bigoplus_{b \in \mathbf{f}_i^n \mathbf{B}} \mathbf{k}b$. We

have

$$f_i^n V \subset \bigoplus_{b \in \mathbf{f}_i^n \mathbf{B}} \mathbf{k}b + f_i^{n+1}V,$$

which easily follows from (i). Then (ii) follows from $(f_i^k V)_\lambda = 0$ for any $\lambda \in P$ and $k \gg 0$.

(iii), (iv) and (v) easily follow from (ii). \square

Let \mathbf{B} be a dual perfect basis of a P -graded \mathbf{k} -vector space V . For $b \in \mathbf{B}$, we define

$$\mathbf{e}_i b = \begin{cases} b' & \text{if } b' \in \mathbf{B} \text{ satisfies } \mathbf{f}_i b' = b, \\ 0 & \text{if there exists no } b' \in \mathbf{B} \text{ such that } \mathbf{f}_i b' = b. \end{cases}$$

We also define the maps $\varepsilon_i, \varphi_i : \mathbf{B} \rightarrow \mathbb{Z} \cup \{-\infty\}$ by

$$\varepsilon_i(b) = \begin{cases} \ell_i(b) & \text{if } i \in I^{\text{re}}, \\ 0 & \text{if } i \in I^{\text{im}}, \end{cases}$$

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle,$$

where the map $\text{wt} : \mathbf{B} \rightarrow P$ is given by $\text{wt}(b) = \mu$ if $b \in \mathbf{B}_\mu$.

Then it is straightforward to verify that $(\mathbf{B}, \mathbf{e}_i, \mathbf{f}_i, \varepsilon_i, \varphi_i, \text{wt})$ is an abstract crystal, which will be called the *dual perfect graph*.

Proposition 4.5. *Let $\mathbf{B}(\lambda)$ be the global basis of $V(\lambda)$ ($\lambda \in P^+$). Then $\mathbf{B}(\lambda)$ is isomorphic to $B(\lambda)$ as an abstract crystal.*

Proof. Recall that $\mathbf{B}(\lambda)$ becomes a dual perfect basis by defining $\mathbf{f}_i G(b) = G(\tilde{f}_i b)$ for $b \in B(\lambda)$. Hence for $b, b' \in B(\lambda)$, we have $\tilde{f}_i b = b'$ if and only if $G(b') = G(\tilde{f}_i b) = \mathbf{f}_i G(b)$, which proves our claim. \square

Remark 4.6. Let \mathfrak{g} be the generalized Kac-Moody algebra associated with a Borcherds-Cartan datum. By taking the classical limit in Proposition 3.3, it follows that every irreducible highest weight \mathfrak{g} -module $V(\lambda)$ ($\lambda \in P^+$) has a dual perfect basis whose dual perfect graph is isomorphic to $B(\lambda)$ as an abstract crystal.

Proposition 4.7.

(i) *The algebra $U_q^-(\mathfrak{g})$ has a dual perfect basis.*

(ii) *The lower global basis $\mathbf{B}(\infty)$ is isomorphic to $B(\infty)$ as an abstract crystal.*

Proof. As in Proposition 4.3 and Proposition 4.5, the lower global basis $\mathbf{B}(\infty)$ satisfies our assertions. \square

5. PROPERTIES OF DUAL PERFECT BASES

Let \mathbf{k} be a field, and \mathbf{B} be a dual perfect basis of $(V, \{f_i\}_{i \in I})$.

For $b \in \mathbf{B}$ and $i \in I$, we set $\mathbf{e}_i^{\text{top}}(b) = \mathbf{e}_i^{\ell_i(b)} b \in \mathbf{B}$. More generally, for a sequence $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ ($m \geq 1$), we set

$$\mathbf{e}_{\mathbf{i}}^{\text{top}} b := \mathbf{e}_{i_m}^{\text{top}} \cdots \mathbf{e}_{i_1}^{\text{top}} b.$$

We also use the notations

$$\mathbf{e}_{\mathbf{i}}^{\mathbf{l}} b := \mathbf{e}_{i_m}^{l_m} \cdots \mathbf{e}_{i_1}^{l_1} b \in \mathbf{B} \sqcup \{0\},$$

$$\mathbf{f}_{\mathbf{i}}^{\mathbf{l}} b := \mathbf{f}_{i_1}^{l_1} \cdots \mathbf{f}_{i_m}^{l_m} b \in \mathbf{B} \sqcup \{0\}$$

for $\mathbf{l} = (l_1, \dots, l_m) \in \mathbb{Z}_{\geq 0}^m$ and set

$$f_{\mathbf{i}}^{\mathbf{l}} := f_{i_1}^{l_1} \cdots f_{i_m}^{l_m}.$$

We say that a sequence $\mathbf{i} = (i_k)_{k \geq 1}$ is *good* if $\{k \in \mathbb{Z}_{>0} \mid i_k = i\}$ is an infinite set for any $i \in I$. We say that a sequence $\mathbf{l} = (l_k)_{k \geq 1}$ of non-negative integers is *good* if $l_k = 0$ for $k \gg 0$. For a good sequence $\mathbf{i} = (i_k)_{k \geq 1}$ in I and a good sequence $\mathbf{l} = (l_k)_{k \geq 1}$ of non-negative integers, we set

$$V^{>\mathbf{l}, \mathbf{i}} = \sum_{k \geq 1} f_{i_1}^{l_1} \cdots f_{i_{k-1}}^{l_{k-1}} f_{i_k}^{1+l_k} V,$$

$$V^{\geq \mathbf{l}, \mathbf{i}} = V^{>\mathbf{l}, \mathbf{i}} + f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} V = \sum_{k=1}^m f_{i_1}^{l_1} \cdots f_{i_{k-1}}^{l_{k-1}} f_{i_k}^{1+l_k} V + f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} V$$

for $m \gg 0$.

Let \prec be the lexicographic ordering on good sequences of integers, namely, $\mathbf{l} = (l_k)_{k \geq 1} \prec \mathbf{l}' = (l'_k)_{k \geq 1}$ if and only if there exists $k \geq 1$ such that $l_s = l'_s$ for any s with $1 \leq s < k$ and $l_k < l'_k$. Then for $\mathbf{l} = (l_k)_{k \geq 1}$ and $\mathbf{l}' = (l'_k)_{k \geq 1}$, we have

$$V^{\geq \mathbf{l}, \mathbf{i}} \subset V^{\geq \mathbf{l}', \mathbf{i}} \quad \text{if } \mathbf{l}' \preceq \mathbf{l},$$

$$V^{>\mathbf{l}, \mathbf{i}} \subset V^{>\mathbf{l}', \mathbf{i}} \quad \text{if } \mathbf{l}' \prec \mathbf{l}.$$

For any $v \in V \setminus \{0\}$ and a good sequence $\mathbf{i} = (i_k)_{k \geq 1}$ in I , we define $L(\mathbf{i}, v) = (l_k)_{k \geq 1}$ to be the largest sequence $\mathbf{l} = (l_k)_{k \geq 1}$ such that $v \in V^{\geq \mathbf{l}}$. Note that such an $L(\mathbf{i}, v)$ exists and is a good sequence (see Proposition 5.1 (ii)). Hence $v \in V^{\geq \mathbf{l}, \mathbf{i}}$ if and only if $\mathbf{l} \preceq L(\mathbf{i}, v)$.

Set $\mathbf{B}_H = \{b \in \mathbf{B} \mid \ell_i(b) = 0 \text{ for all } i \in I\}$. For a good sequence $\mathbf{i} = (i_k)_{k \geq 1}$, we set $\mathbf{e}_{\mathbf{i}}^{\text{top}} = \mathbf{e}_{i_m}^{\text{top}} \cdots \mathbf{e}_{i_1}^{\text{top}} b$ for $m \gg 0$. Note that it does not depend on $m \gg 0$ and belongs to \mathbf{B}_H (see Proposition 5.1 (ii)).

Proposition 5.1. *Let $\mathbf{i} = (i_k)_{k \geq 1}$ be a good sequence in I .*

- (i) *For any $b \in \mathbf{B}$ and a sequence $\mathbf{l} = (l_1, \dots, l_m)$ of non-negative integers, there exists $c \in \mathbf{k}^\times$ such that*

$$f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} b - c \mathbf{f}_{i_1}^{l_1} \cdots \mathbf{f}_{i_m}^{l_m} b \in \sum_{k=1}^m f_{i_1}^{l_1} \cdots f_{i_k}^{1+l_k} V.$$

- (ii) *For each $b \in \mathbf{B}$, define a sequence $(b_k)_{k \geq 0}$ by*

$$b_0 = b, \quad b_k = \mathbf{e}_{i_k}^{\text{top}} b_{k-1} \quad \text{for } k \geq 1,$$

and let $(L_k)_{k \geq 1}$ be the sequence of non-negative integers given by

$$L_k = l_{i_k}(b_{k-1}) \quad \text{for } k \geq 1.$$

Then we have

- (a) $(L_k)_{k \geq 1}$ is a good sequence,
(b) $b_k \in B_H$ for $k \gg 0$,
(c) $L(\mathbf{i}, b) = (L_k)_{k \geq 1}$.
(iii) *For any good sequence $\mathbf{l} = (l_k)_{k \geq 1}$ of non-negative integers, we have*

$$(5.1) \quad V^{\geq \mathbf{l}, \mathbf{i}} = \sum_{\{b \in \mathbf{B} \mid \mathbf{l} \leq L(\mathbf{i}, b)\}} \mathbf{k} b,$$

$$(5.2) \quad V^{> \mathbf{l}, \mathbf{i}} = \sum_{\{b \in \mathbf{B} \mid \mathbf{l} < L(\mathbf{i}, b)\}} \mathbf{k} b.$$

- (iv) *For any good sequence $\mathbf{l} = (l_k)_{k \geq 1}$ of non-negative integers, we have an injective map*

$$(5.3) \quad \mathbf{e}^{\mathbf{l}}: \{b \in \mathbf{B} \mid L(\mathbf{i}, b) = \mathbf{l}\} \rightarrow \mathbf{B}_H.$$

Proof. Let us first prove (i) by induction on m . The $m = 1$ case follows from Lemma 4.4 (i). Assume that $m > 1$. Set $b_1 = \mathbf{f}_{i_m}^{l_m} b$. Then applying the induction hypothesis to b_1 , there exists $c \in \mathbf{k}^\times$ such that

$$f_{i_1}^{l_1} \cdots f_{i_{m-1}}^{l_{m-1}} b_1 - c \mathbf{f}_{i_1}^{l_1} \cdots \mathbf{f}_{i_{m-1}}^{l_{m-1}} b_1 \in \sum_{k=1}^{m-1} f_{i_1}^{l_1} \cdots f_{i_k}^{1+l_k} V.$$

By Lemma 4.4 (i), there exists $c' \in \mathbf{k}^\times$ such that $f_{i_m}^{l_m} b - c' b_1 \in f_{i_m}^{1+l_m} V$. Hence we have

$$\begin{aligned} f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} b - c c' \mathbf{f}_{i_1}^{l_1} \cdots \mathbf{f}_{i_m}^{l_m} b &= f_{i_1}^{l_1} \cdots f_{i_{m-1}}^{l_{m-1}} (f_{i_m}^{l_m} b - c' b_1) \\ &+ c' (f_{i_1}^{l_1} \cdots f_{i_{m-1}}^{l_{m-1}} b_1 - c \mathbf{f}_{i_1}^{l_1} \cdots \mathbf{f}_{i_{m-1}}^{l_{m-1}} b_1) \in \sum_{k=1}^m f_{i_1}^{l_1} \cdots f_{i_k}^{1+l_k} V. \end{aligned}$$

Next we shall show (ii) (a) and (ii) (b). Since $\text{wt}(b) - \sum_{k=1}^m L_k \alpha_{i_k} \in \text{wt}(V)$ for any m , we have $L_k = 0$ for $k \gg 0$. Hence b_k does not depend on $k \gg 0$. Thus $\ell_{i_{k+1}}(b_k) = 0$ for $k \gg 0$ which implies (ii) (b).

To prove (ii) (c) and (iii), let $\mathbf{L} = (L_k)_{k \geq 1}$ be the sequence in (ii) which is uniquely determined for each $b \in \mathbf{B}$. For $m \in \mathbb{Z}_{\geq 0}$, set $\tilde{\mathbf{L}}_m(\mathbf{i}, b) = (L_1, \dots, L_m)$. We first observe that for any sequence $\mathbf{l} = (l_1, \dots, l_m)$ with $\mathbf{l} \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, b)$, we have

$$(5.4) \quad b \in \sum_{k=1}^{m-1} f_{i_1}^{l_1} \cdots f_{i_{k-1}}^{l_{k-1}} f_{i_k}^{1+l_k} V + f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} V \quad \text{if } (l_1, \dots, l_m) \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, b),$$

which immediately follows from (i).

Now we shall show

$$(5.5) \quad \sum_{k=1}^{m-1} f_{i_1}^{l_1} \cdots f_{i_{k-1}}^{l_{k-1}} f_{i_k}^{1+l_k} V + f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} V = \sum_{\{b \in \mathbf{B} \mid (l_1, \dots, l_m) \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, b)\}} \mathbf{k} b.$$

We have already seen that the right hand side is contained in the left hand side. Let us show the converse inclusion by induction on m . In order to see this, it is enough to show that

$$f_{i_1}^{l_1} \cdots f_{i_m}^{l_m} V \subset \sum_{\{b \in \mathbf{B} \mid (l_1, \dots, l_m) \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, b)\}} \mathbf{k} b.$$

Set $\mathbf{i}' = (i_2, i_3, \dots)$. Then the induction hypothesis implies

$$f_{i_2}^{l_2} \cdots f_{i_m}^{l_m} V \subset \sum_{\{b \in \mathbf{B} \mid (l_2, \dots, l_m) \preceq \tilde{\mathbf{L}}_{m-1}(\mathbf{i}', b)\}} \mathbf{k} b.$$

Hence we have reduced (5.5) to

$$f_{i_1}^{l_1} b_0 \in \sum_{\{b \in \mathbf{B} \mid (l_1, \dots, l_m) \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, b)\}} \mathbf{k} b$$

if $(l_2, \dots, l_m) \preceq \tilde{\mathbf{L}}_{m-1}(\mathbf{i}', b_0)$, which follows from the fact that $f_{i_1}^{l_1} b_0 \in \mathbf{k} \mathbf{f}_{i_1}^{l_1} b_0 + \sum_{\ell_i(b) > l_1} \mathbf{k} b$ and $(l_1, \dots, l_m) \preceq \tilde{\mathbf{L}}_m(\mathbf{i}, \mathbf{f}_{i_1}^{l_1} b_0)$. Thus the proof of (5.5) is complete.

Now, (ii) (c) follows from (5.5), and then (5.1) is nothing but (5.5) for $m \gg 0$. Equality (5.2) follows easily from (5.1).

In order to prove (iv), observe that $\mathbf{e}^{\mathfrak{l}}(b) = \mathbf{e}_i^{\text{top}}(b)$, where $\mathbf{e}^{\mathfrak{l}}$ is the map defined in (5.3) and $L(\mathbf{i}, b) = \mathfrak{l}$. Since $\mathbf{e}^{\mathfrak{l}}$ has a left inverse $\mathbf{f}_i^{\mathfrak{l}}|_{\mathbf{B}_H}$, we conclude that $\mathbf{e}^{\mathfrak{l}}$ is injective. \square

The following corollary easily follows from the preceding proposition.

Corollary 5.2. *Let $\mathbf{i} = (i_k)_{k \geq 1}$ be a good sequence in I and let $\mathfrak{l} = (l_k)_{k \geq 1}$ be a good sequence of non-negative integers. Denote by $p_{\mathbf{i}, \mathfrak{l}}: V^{\geq \mathfrak{l}, \mathbf{i}} \rightarrow V^{\geq \mathfrak{l}, \mathbf{i}}/V^{> \mathfrak{l}, \mathbf{i}}$ the canonical projection and set $\mathbf{B}_{\mathbf{i}, \mathfrak{l}} = \{b \in \mathbf{B} \mid L(\mathbf{i}, b) = \mathfrak{l}\}$.*

Then the image $p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}_{\mathbf{i}, \mathfrak{l}})$ is a basis of $V^{\geq \mathfrak{l}, \mathbf{i}}/V^{> \mathfrak{l}, \mathbf{i}}$. Moreover, $\mathbf{k}^\times p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}_{\mathbf{i}, \mathfrak{l}})$ is equal to $\mathbf{k}^\times (p_{\mathbf{i}, \mathfrak{l}}(\mathbf{f}_i^{\mathfrak{l}} \mathbf{B}_H) \setminus \{0\})$.

Here, for a subset S of a \mathbf{k} -vector space V , we use the notation

$$\mathbf{k}^\times S := \{\mathbf{k}^\times s \mid s \in S\}.$$

Note that Proposition 5.1 (iii) with $\mathfrak{l} = (0, 0, \dots)$ implies that

$$\sum_{i \in I} f_i V = \bigoplus_{b \in \mathbf{B} \setminus \mathbf{B}_H} \mathbf{k} b.$$

Hence we conclude

Lemma 5.3. *Set $V_H := V / (\sum_{i \in I} f_i V)$ and let $p_H: V \twoheadrightarrow V_H$ be the canonical projection. Then $p_H: \mathbf{B}_H \rightarrow V_H$ is injective and $p_H(\mathbf{B}_H)$ is a basis of V_H .*

6. UNIQUENESS OF DUAL PERFECT GRAPHS

The purpose of this section is to prove that all the dual perfect graphs of a given dual perfect space are isomorphic as abstract crystals.

Theorem 6.1. *Let $(V, \{f_i\}_{i \in I})$ be a dual perfect space and let \mathbf{B} and \mathbf{B}' be its dual perfect bases. Assume that $p_H(\mathbf{B}_H) = p_H(\mathbf{B}'_H)$.*

Then there is a crystal isomorphism $\psi: \mathbf{B} \xrightarrow{\sim} \mathbf{B}'$ such that $p_H(b) = p_H(\psi(b))$ for all $b \in \mathbf{B}_H$. Moreover, for any $b \in \mathbf{B}$ and a good sequence \mathbf{i} in I , we have $L(\mathbf{i}, b) = L(\mathbf{i}, \psi(b))$ and $p_H(\mathbf{e}_i^{\text{top}} b) = p_H(\mathbf{e}_i^{\text{top}} \psi(b)) \in p_H(\mathbf{B}_H)$.

Proof. For a good sequence $\mathbf{i} = (i_k)_{k \geq 1}$ in I and a good sequence $\mathfrak{l} = (l_k)_{k \geq 1}$, let $p_{\mathbf{i}, \mathfrak{l}}: V^{\geq \mathfrak{l}, \mathbf{i}} \rightarrow V^{\geq \mathfrak{l}, \mathbf{i}}/V^{> \mathfrak{l}, \mathbf{i}}$ be the canonical projection. Set $\mathbf{B}_{\mathbf{i}, \mathfrak{l}} = \{b \in \mathbf{B} \mid L(\mathbf{i}, b) = \mathfrak{l}\}$ and define $\mathbf{B}'_{\mathbf{i}, \mathfrak{l}}$ in a similar manner. Then by Corollary 5.2 and Lemma 5.3, we have

$$\mathbf{k}^\times p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}_{\mathbf{i}, \mathfrak{l}}) = \mathbf{k}^\times p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}'_{\mathbf{i}, \mathfrak{l}})$$

and both $p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}_{\mathbf{i}, \mathfrak{l}})$ and $p_{\mathbf{i}, \mathfrak{l}}(\mathbf{B}'_{\mathbf{i}, \mathfrak{l}})$ are bases of $V^{\geq \mathfrak{l}, \mathbf{i}}/V^{> \mathfrak{l}, \mathbf{i}}$. Hence for any $b \in \mathbf{B}$, there exists $b' \in \mathbf{B}'$ such that

$$(6.1) \quad L(\mathbf{i}, b) = L(\mathbf{i}, b') \text{ and } b - cb' \in V^{> L(\mathbf{i}, b), \mathbf{i}} \text{ for some } c \in \mathbf{k}^\times.$$

To prove our claim, it is enough to show that for another choice of a good sequence \mathbf{i}' , (6.1) holds with the same b' . Set $v = b - cb'$. Since $v \in V^{>L(\mathbf{i},b),\mathbf{i}}$, v is a linear combination of $\mathbf{B} \setminus \{b\}$. Set $\mathbf{l} = L(\mathbf{i}', b)$ and $\mathbf{l}' = L(\mathbf{i}', b')$. Then $b \in V^{\geq \mathbf{l}, \mathbf{i}'}$ and $b' \in V^{\geq \mathbf{l}', \mathbf{i}'}$. Assume that $\mathbf{l} \prec \mathbf{l}'$. Since $b' \in V^{> \mathbf{l}, \mathbf{i}'}$, we have $v - b \in V^{> \mathbf{l}, \mathbf{i}'}$. Hence $v - b$ is a linear combination of $\mathbf{B} \setminus \{b\}$, which is a contradiction to the fact that v is a linear combination of $\mathbf{B} \setminus \{b\}$. Hence we conclude that $\mathbf{l}' \preceq \mathbf{l}$. Similarly, we have $\mathbf{l} \preceq \mathbf{l}'$. Hence we obtain $\mathbf{l} = \mathbf{l}'$. It follows that both $p_{\mathbf{i}', \mathbf{l}}(b)$ and $p_{\mathbf{i}', \mathbf{l}}(b')$ belong to $\mathbf{k}^\times p_{\mathbf{i}', \mathbf{l}}(\mathbf{B}_{\mathbf{i}, \mathbf{l}})$. If $\mathbf{k}^\times p_{\mathbf{i}', \mathbf{l}}(b) \neq \mathbf{k}^\times p_{\mathbf{i}', \mathbf{l}}(b')$, then $v - b$ is a linear combination of $\mathbf{B} \setminus \{b\}$, which is a contradiction. Hence $\mathbf{k}^\times p_{\mathbf{i}', \mathbf{l}}(b) = \mathbf{k}^\times p_{\mathbf{i}', \mathbf{l}}(b')$ and our assertion follows. \square

Corollary 6.2. *Let $U_q(\mathfrak{g})$ be the quantum generalized Kac-Moody algebra associated with a Borcherds-Cartan datum and let $V_q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$ -module with $\lambda \in P^+$. Then the following statements hold.*

- (i) *Every dual perfect graph of $V_q(\lambda)$ is isomorphic to $B(\lambda)$ as an abstract crystal.*
- (ii) *Every dual perfect graph of $U_q(\mathfrak{g})$ is isomorphic to $B(\infty)$ as an abstract crystal.*

Remark 6.3. Theorem 6.1 also shows that every dual perfect graph of $V(\lambda)$ over a generalized Kac-Moody algebra \mathfrak{g} is isomorphic to $B(\lambda)$ as an abstract crystal.

7. PERFECT BASES AND DUAL PERFECT BASES

Now we will prove that the isomorphism classes of finitely generated graded projective modules over R and R^λ form a dual perfect basis of $\mathbb{Q}(q) \otimes_{\mathbb{Q}[q, q^{-1}]} [\text{Proj}(R)]$ and $\mathbb{Q}(q) \otimes_{\mathbb{Q}[q, q^{-1}]} [\text{Proj}(R^\lambda)]$, respectively. We first recall the definition of *perfect basis*.

Let $V = \bigoplus_{\mu \in P} V_\mu$ be a P -graded \mathbf{k} -vector space with a family of endomorphisms $\{e_i\}_{i \in I}$ satisfying the following conditions.

- (i) There exist finitely many elements $\lambda_1, \dots, \lambda_k \in P$ such that $\text{wt}(V) \subset \bigcup_{j=1}^k (\lambda_j - Q^+)$.
- (ii) $e_i(V_\mu) \subset V_{\mu + \alpha_i}$ for any $i \in I$ and $\mu \in P$.

Set $\ell_i^\vee(v) := \max\{n \in \mathbb{Z}_{\geq 0} \mid e_i^n v \neq 0\} = \min\{n \in \mathbb{Z}_{\geq 0} \mid e_i^{n+1} v = 0\}$ for $v \in V \setminus \{0\}$.

Definition 7.1. A basis \mathbb{B} of V is said to be *perfect* if

- (1) $\mathbb{B} = \bigoplus_{\mu \in P} \mathbb{B}_\mu$, where $\mathbb{B}_\mu := \mathbb{B} \cap V_\mu$,
- (2) for any $i \in I$, there exists a map $\mathbf{E}_i: \mathbb{B} \rightarrow \mathbb{B} \cup \{0\}$ such that for any $b \in \mathbb{B}$, we have
 - (a) if $\ell_i^\vee(b) = 0$, then $\mathbf{E}_i(b) = 0$,
 - (b) if $\ell_i^\vee(b) > 0$, then $\mathbf{E}_i(b) \in \mathbb{B}$ and

$$e_i(b) - c\mathbf{E}_i(b) \in \ker e_i^{\ell_i^\vee(b)-1} \quad \text{for some } c \in \mathbf{k}^\times,$$

- (3) if $\mathbf{E}_i(b) = \mathbf{E}_i(b') \neq 0$ for $b, b' \in \mathbb{B}$, then $b = b'$.

It was shown in [7] and [8] that every $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} and the negative half $U_q^-(\mathfrak{g})$ have perfect bases.

For a perfect basis \mathbb{B} , we define a map $\mathbf{F}_i: \mathbb{B} \rightarrow \mathbb{B} \cup \{0\}$ by

$$\mathbf{F}_i(b) = \begin{cases} b' & \text{if } \mathbf{E}_i(b') = b, \\ 0 & \text{if } b \notin \mathbf{E}_i\mathbb{B}. \end{cases}$$

Let us recall the following lemma.

Lemma 7.2 ([1]). *Let \mathbb{B} be a perfect basis of $(V, \{e_i\}_{i \in I})$. Then*

(i) *for any $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\ker e_i^n = \bigoplus_{b \in \mathbb{B}, \ell_i^\vee(b) < n} \mathbf{k}b.$$

(ii) $\ell_i^\vee(\mathbf{E}_i b) = \ell_i^\vee(b) - 1$ *for any $b \in \mathbb{B}$ such that $\ell_i^\vee(b) > 0$.*

(iii) *for $b \in \mathbb{B}$ and $i \in I$, we have $b \in \mathbf{E}_i\mathbb{B}$ if and only if $b \in e_i V + \ker e_i^{\ell_i^\vee(b)}$.*

(iv) *for any $n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, the image of $\{b \in \mathbb{B} \mid \ell_i^\vee(b) = n\}$ is a basis of $\ker e_i^{n+1} / \ker e_i^n$.*

Let $(V, \{f_i\}_{i \in I})$ be a pre-dual perfect space such that $\dim V_\lambda < \infty$ for any $\lambda \in P$. We set

$$(7.1) \quad (V^\vee)_\lambda = \text{Hom}_{\mathbf{k}}(V_\lambda, \mathbf{k}) \quad \text{for any } \lambda \in P, \text{ and } V^\vee = \bigoplus_{\lambda \in P} (V^\vee)_\lambda.$$

Let $\langle \bullet, \bullet \rangle : V \times V^\vee \rightarrow \mathbf{k}$ be the canonical pairing. We define $e_i: V^\vee \rightarrow V^\vee$ as the transpose of f_i , so that we have $\langle u, e_i v \rangle = \langle f_i u, v \rangle$ for any $u \in V$ and $v \in V^\vee$.

Proposition 7.3. *Let $(V, \{f_i\}_{i \in I})$ be a pre-dual perfect space such that $\dim V_\lambda < \infty$ for any $\lambda \in P$. Let \mathbf{B} be a basis of V and let $\mathbf{B}^\vee \subset V^\vee$ be the dual basis of \mathbf{B} . Then*

- (i) \mathbf{B} *is a dual perfect basis if and only if \mathbf{B}^\vee is a perfect basis.*
- (ii) *Assume that \mathbf{B} is a dual perfect basis. Denoting the canonical isomorphism $\mathbf{B} \xrightarrow{\sim} \mathbf{B}^\vee$ by $b \mapsto b^\vee$, we have*

$$\ell_i(b) = \ell_i^\vee(b^\vee), \quad \text{and} \quad (\mathbf{e}_i b)^\vee = \mathbf{E}_i(b^\vee) \quad \text{for all } b \in \mathbf{B} \text{ and } i \in I.$$

Here we understand $0^\vee = 0$.

Proof. Assume first that \mathbf{B}^\vee is a perfect basis. Then, $\ker e_i^n = \bigoplus_{b^\vee \in \mathbf{B}^\vee, \ell_i^\vee(b^\vee) < n} \mathbf{k}b^\vee$.

Since $f_i^n V$ coincides with the orthogonal complement $(\ker e_i^n)^\perp := \{u \in V \mid \langle u, \ker e_i^n \rangle = 0\}$ of $\ker e_i^n$, we have $f_i^n V = \bigoplus_{\ell_i^\vee(b^\vee) \geq n} \mathbf{k}b$. Hence $b \in f_i^n V$ if and only if $\ell_i^\vee(b^\vee) \geq n$.

Therefore we have $\ell_i(b) = \ell_i^\vee(b^\vee)$, and

$$f_i^n V = \bigoplus_{\ell_i(b) \geq n} \mathbf{k}b.$$

We define $\mathbf{f}_i: \mathbf{B} \rightarrow \mathbf{B} \cup \{0\}$ by $(\mathbf{f}_i b)^\vee = \mathbf{F}_i(b^\vee)$. Let $b \in \mathbf{B}$ with $\ell_i(b) = n$. We shall show that $f_i b - c \mathbf{f}_i b \in f_i^{n+2} V$ for some $c \in \mathbf{k}^\times$.

Recall that the image of $\mathbf{B}_n^\vee := \{b^\vee \in \mathbf{B}^\vee \mid \ell_i^\vee(b^\vee) = n\}$ forms a basis of $\ker e_i^{n+1} / \ker e_i^n$. Since $\ker e_i^{n+1} / \ker e_i^n$ is isomorphic to the dual $(f_i^n V / f_i^{n+1} V)^\vee$ of $f_i^n V / f_i^{n+1} V$, the image of \mathbf{B}_n forms a basis of $f_i^n V / f_i^{n+1} V$ dual to \mathbf{B}_n^\vee . By the hypothesis, the map $e_i: \ker e_i^{n+2} / \ker e_i^{n+1} \rightarrow \ker e_i^{n+1} / \ker e_i^n$ gives an injection

$$\mathbf{E}_i: \{b^\vee \in \mathbf{B}_{n+1}^\vee \mid \mathbf{E}_i(b^\vee) \neq 0\} \rightarrow \mathbf{B}_n^\vee.$$

Hence $f_i: f_i^n V / f_i^{n+1} V \rightarrow f_i^{n+1} V / f_i^{n+2} V$ sends $\{b \in \mathbf{B}_n \mid \mathbf{F}_i(b^\vee) \neq 0\}$ to \mathbf{B}_{n+1} up to a constant multiple, and sends $\{b \in \mathbf{B}_n \mid \mathbf{F}_i(b^\vee) = 0\}$ to $\{0\}$.

As a conclusion, for any $b \in \mathbf{B}_n$, we have $f_i b \equiv c \mathbf{f}_i b \pmod{f_i^{n+2} V}$ for some $c \in \mathbf{k}^\times$. Thus \mathbf{B} is a dual perfect basis.

The converse can be proved in a similar manner. \square

Recall that the Khovanov-Lauda-Rouquier algebra R and its cyclotomic quotient R^λ provide a categorification of $U_q(\mathfrak{g})$ and V_q^λ , respectively. That is, we have

$$\begin{aligned} U_q^-(\mathfrak{g}) &\cong \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R)] \cong \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R)], \\ V_q(\lambda) &\cong \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R^\lambda)] \cong \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R^\lambda)]. \end{aligned}$$

It was shown in [8, 13] that the isomorphism classes of finite-dimensional graded irreducible modules form a perfect basis of $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R)]$ and $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Rep}(R^\lambda)]$, respectively. Since the isomorphism classes of finite-dimensional graded irreducible R -modules and those of finitely generated graded projective indecomposable R -modules are dual to each other with respect to the perfect pairing given by

$$\begin{aligned} [\text{Proj}(R)] \times [\text{Rep}(R)] &\longrightarrow \mathbb{A}, \\ ([P], [M]) &\longmapsto \dim_q \text{Hom}_R(P, M), \end{aligned}$$

the following proposition follows immediately.

Proposition 7.4. *The isomorphism classes of finitely generated graded projective indecomposable modules form a dual perfect basis of $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R)]$ and $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} [\text{Proj}(R^\lambda)]$.*

REFERENCES

- [1] A. Berenstein, D. Kazhdan, *Geometric and unipotent crystals. II: From unipotent bicrystals to crystal bases*, Contemp. Math. **433** (2007), 13–88.
- [2] J. Hong, S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics **42**, Amer. Math. Soc., Providence, 2002.
- [3] K. Jeong, S.-J. Kang, M. Kashiwara, *Crystal bases for quantum generalized Kac-Moody algebras*, Proc. Lond. Math. Soc. **90** (3) (2005), 395–438.
- [4] S.-J. Kang, M. Kashiwara, *Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras*, Invent. Math. **190** (2012), 699–742.

- [5] S.-J. Kang, M. Kashiwara, S.-j. Oh, *Categorification of highest weight modules over quantum generalized Kac-Moody algebras*, Moscow Math. J. **13** (2013), 315–343.
- [6] S.-J. Kang, M. Kashiwara, E. Park, *Geometric realization of Khovanov-Lauda-Rouquier algebras associated with Borchers-Cartan data*, Proc. London Math. Soc. (3) **107** (2013), 907–231.
- [7] S.-J. Kang, S.-j. Oh, E. Park, *Perfect bases for integrable modules over generalized Kac-Moody algebras*, Algebr. Represent. Theory **14** (2011), 571–587.
- [8] ———, *Categorification of quantum generalized Kac-Moody algebras and crystal bases*, Int. J. Math. **23** (2012), 1250116.
- [9] ———, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [10] ———, *Global crystal bases of quantum groups*, Duke math. J. **69** (1993), 455–485.
- [11] M. Khovanov, A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.
- [12] ———, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), 2685–2700.
- [13] A. Lauda, M. Vazirani, *Crystals from categorified quantum groups*, Adv. Math. **228** (2011), 803–861.
- [14] R. Rouquier, *2 Kac-Moody algebras*, arXiv:0812.5023.
- [15] ———, *Quiver Hecke algebras and 2-Lie algebras*, arXiv:1112.3619v1.
- [16] M. Varagnolo, E. Vasserot, *Canonical bases and KLR algebras*, J. reine angew. Math. **659** (2011), 67–100.

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